

# $\mathcal{N} = 1$ Field Theories and Fluxes in IIB String Theory

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## Abstract

Deformation of  $\mathcal{N} = 2$  quiver gauge theories by adjoint masses leads to fixed manifolds of  $\mathcal{N} = 1$  superconformal field theories. We elaborate on the role of the complex three-form flux in the IIB duals to these fixed point theories, primarily using field theory techniques. We study the moduli space at a fixed point and find that it is either the two (complex) dimensional ALE space or three-dimensional generalized conifold, depending on the type of three-form flux that is present. We describe the exactly marginal operators that parameterize the fixed manifolds and find the operators which preserve the dimension of the moduli space. We also study deformations by arbitrary superpotentials  $W(\Phi_i)$  for the adjoints. We invoke the  $a$ -theorem to show that there are no dangerously irrelevant operators like  $\text{Tr } \Phi_i^{k+1}$ ,  $k > 2$  in the  $\mathcal{N} = 2$  quiver gauge theories. The moduli space of the IR fixed point theory generally contains orbifold singularities if  $W(\Phi_i)$  does not give a mass to the adjoints. Finally we examine some nonconformal  $\mathcal{N} = 1$  quiver theories. We find evidence that the moduli space at the endpoint of a Seiberg duality cascade is always a three-dimensional generalized conifold. In general, the low-energy theory receives quantum corrections. In several non-cascading theories we find that the moduli space is a generalized conifold realized as a monodromic fibration.

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# 1 Introduction

One of the biggest successes of string theory in the last 10 years has been the development of the correspondence between gauge theories and gravity. One aspect of this correspondence is the geometric engineering of gauge theories by considering string theory backgrounds in certain limits. In this paper, we will discuss various aspects of some of the  $\mathcal{N} = 1$  gauge theories in four dimensions that can be obtained by deforming the theories on D-branes at orbifold singularities in IIB string theory.

The AdS/CFT correspondence [1,2] provides one concrete route toward the theories we study in this paper. One begins with D3-branes at the ALE orbifold  $\mathbb{C}^2/\Gamma$ ,  $\Gamma \subset SU(2)$ , whose near horizon limit is given by IIB supergravity in the  $AdS_5 \times S^5/\Gamma$  background [3]. The dual conformal field theory in this case is an  $\mathcal{N} = 2$  quiver gauge theory [4]. Deformations of this theory by relevant operators will drive the theory to new conformal fixed points. One class of relevant operators are certain mass terms for the adjoint chiral fields [5,6], which drive the theory to fixed points with  $\mathcal{N} = 1$  superconformal invariance. These fixed point theories are parameterized by superpotential couplings  $h_i$ . Generally, these deformations lead to manifolds of fixed points parameterized by the ratios  $h_i/h_j$ , so the manifold of fixed points is a complex projective space  $\mathbb{P}^{n-1}$  [7].

The geometry dual to these fixed points is that of  $AdS_5 \times X_5$ , where  $X_5$  is the base of a generalized conifold [5,6,8–10]. By a generalized conifold, we mean the three complex-dimensional manifold obtained by fibering an ALE space over the complex line [11]. In the field theory, this fibration arises directly from adding the mass terms. The orbifold singularities of the ALE space are replaced by conifold singularities at points on the line. The masses themselves correspond to complex structure moduli of the generalized conifold.

The geometry is, however, not the complete story, as a mass deformation with  $\sum_i h_i^{-1} \neq 0$  also introduces flux for the complex IIB 3-form field strength. In the presence of this flux, the metric on the generalized conifold will differ from the Ricci-flat one. The relation of these fluxes to geometric deformations was described in a framework of 5D gauged supergravity in [7]. There the map between mass terms and fields in the untwisted and twisted sectors of the orbifold string theory was examined. The symmetries of the manifold of fixed points in the dual gauge theory were related to the duality symmetries of the orbifold theory.

Our motivation for the present work was to shed further light on this larger picture of 3-form flux in these theories. In Section 3, we review the results of [7] on the correspondence between untwisted and twisted sector fields and the relevant operators which drive the RG flows to the manifold of fixed points. Some solutions are known for the supergravity duals to the fixed points generated by purely twisted deformations.

For  $A_1$ , the gravity dual for a purely twisted deformation is just IIB on the conifold [5]. More generally, for the  $A_{n-1}$  case, there is some information for the duals to purely twisted deformations [6]. In the case of a purely untwisted deformation, the solution is an orbifold of the squashed sphere with 3-form flux solution of [12, 13]. However very little detail is available when both twisted and untwisted deformations are present. Hopefully, our field theory results can prove useful in shedding light on the structure of the gravity duals of these fixed points.

In Section 4, we give a purely field theoretic analysis of the fixed points. One new tool we employ is the  $a$  maximization technique of [14] to compute the exact  $\mathcal{N} = 1$  superconformal  $U(1)_R$  R-charges. We then compute the moduli space of the scalar fields in the gauge theory at the  $\mathcal{N} = 1$  fixed points. The primary result is that, whenever the mass deformation includes a deformation corresponding to the untwisted sector, the moduli space is just two complex-dimensional. Namely, it is the ALE orbifold, possibly resolved by D-terms. The existence of a fixed point requires that F-terms in the quiver theory are zero. When the mass deformation is purely in the twisted sector, corresponding to the condition that  $\sum_i h_i^{-1} = 0$ , we recover the generalized conifold as the moduli space, in agreement with earlier works [6, 9]. We refer to the surface  $\sum_i h_i^{-1} = 0$  on the manifold of fixed points as the “conifold subspace.” This submanifold has the geometry of a  $\mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$ .

The fact that untwisted mass deformations reduce the dimension of the moduli space of scalars is intimately related to the presence of 3-form flux in the string dual. It is known that this 3-form flux can generate a non-zero potential for a probe D3-brane in this geometry [15]. The moduli space of the gauge theory is located at the minimum of this potential. This corresponds to a point on the complex line used to realize the generalized conifold as an ALE fibration, so the moduli space reduces to just the ALE fiber. The gauge theory results suggest that this ALE space has a resolution, but that the untwisted 3-form flux presents an obstruction to complex structure deformations.

Another main result of the paper is Section 5, where we discuss the exactly marginal operators that parameterize the manifolds of fixed points. We find that a general exactly marginal perturbation will take a point on the conifold subspace off to a point where the dual string background has nonzero 3-form flux. From the moduli space perspective, the generic perturbation lifts one flat (complex) direction, reducing the dimension of moduli space. We find the form of the operators which preserve the conifold subspace, which depend on the initial position on the fixed manifold.

In Section 6, we analyze deformations of the quiver gauge theories by arbitrary polynomial superpotentials for the adjoint chiral fields,  $W(\Phi_i)$ , following [9, 10]. These deformations include operators which are irrelevant at the  $\mathcal{N} = 2$  UV fixed point. Our main result is that deformations by irrelevant operators do not lead to new conformal

fixed points. We show this by demonstrating that these fixed points, if they existed, would lead to violations of the conjectured  $a$ -theorem [16, 17]. Specifically, using a result of [10] on the central charge of the candidate fixed points for monomial deformations  $\Phi_i^{k+1}$ , we show that the flows away from these point generated by relevant perturbations  $\Phi_i^{k'+1}$ ,  $k' < k$ , would violate the  $a$ -theorem. We then compute the moduli spaces for these theories by assuming that in the IR the theory is sitting at a critical point of  $W(\Phi_i)$ . Then we can use the effective mass term for perturbations around the critical point to recover the moduli spaces with data specified by the effective masses.

Nonconformal field theories can also be studied by adding fractional branes to the string backgrounds describing the conformal field theory [18–21, 9, 10]. In these nonconformal field theories, the superpotential generally receives corrections. Correspondingly, the moduli space is a deformed version of the moduli space of the related conformal field theory. In Section 7, we generalize some of the field theory discussion of [21] to theories which arise from untwisted deformations of  $\mathcal{N} = 2$  theories. We discuss the RG flow and Seiberg duality.

These theories are characterized by couplings  $h_i$  in a quartic superpotential. In analogy with the conformal theories, when  $\sum_i h_i^{-1} \neq 0$ , the moduli space of scalars is a two complex-dimensional ALE space, whereas when  $\sum_i h_i^{-1} = 0$ , it is a generalized conifold. We find an indication that, in theories which undergo a duality cascade, the cascade maps the theory onto a new theory which is in the conifold subspace. When  $\sum_i h_i^{-1} \neq 0$ , the moduli space is growing an extra dimension at the end of the cascade. However, quantum corrections to the superpotential could change this result.

We also study some examples in which the corrections to the superpotential are known. These theories are too simple to undergo a duality cascade. The quantum corrections lead to a complex deformation of the ALE or generalized conifold moduli space. In particular, the deformation of the generalized conifold leads to a monodromic fibration structure.

Section 2 contains an introduction to  $\mathcal{N} = 2$  quiver gauge theories, in order to set up notation and some of the computations made in later sections. We conclude with a discussion of our results and interesting future directions of research.

## 2 Review of $\mathcal{N} = 2$ Quiver Gauge Theories

The  $\mathcal{N} = 2$  quiver gauge theories can be easily described using the language of  $\mathcal{N} = 1$  superfields. The gauge group  $G$  is a product  $G = \times_{i=1}^n G_i$  of Lie groups  $G_i$ . The  $\mathcal{N} = 2$  vector multiplets contribute matter consisting of  $\mathcal{N} = 1$  chiral fields  $\Phi_i$  in the adjoint of  $G_i$ . For each factor of the gauge group, we add a vertex to the quiver diagram. Additional matter can come in the form of  $\mathcal{N} = 2$  hypermultiplets, which decompose

into pairs of  $\mathcal{N} = 1$  chiral fields  $A_{ij}$  and  $B_{ji}$ . We will consider the case that the hypermultiplets lie in bifundamental representations of the gauge group, *i. e.*,  $A_{ij}$  is in the  $(\mathbf{N}_i, \bar{\mathbf{N}}_j)$  and  $B_{ji}$  is in the  $(\bar{\mathbf{N}}_i, \mathbf{N}_j)$ . For each field  $A_{ij}$  we draw an oriented line from the vertex  $i$  to the vertex  $j$ , while for each  $B_{ji}$  we draw an oriented line from  $j$  to  $i$ . It is possible to have several “flavors” of fields connecting two vertices with the same orientation. To avoid cluttering notation, we will resist adding flavor indices to our discussion.

There is a unique renormalizable superpotential allowed in these theories,

$$W_{\text{tree}} = \sum_{i \neq j} \lambda_i \text{Tr}_{\mathbf{N}_i} a_{ij} \Phi_i (A_{ij} B_{ji} - B_{ij} A_{ji}), \quad (2.1)$$

where  $a_{ij} \in \{0, 1\}$  are the elements of the (symmetric) adjacency matrix of the quiver. The theory with this superpotential admits  $\mathcal{N} = 2$  SUSY when the couplings  $\lambda_i = q_i g_i$ , where  $g_i$  is the gauge coupling of  $G_i$  and  $q_i$  is the charge of  $A_{ij}$  under  $G_i$ . There is an  $SU(2) \times U(1)_R$  R-symmetry under which  $\Phi_i$  are in the  $\mathbf{1}_2$  and the pairs  $(A_{ij}, B_{ji})$  correspond to a  $\mathbf{2}_0$ . However, this  $SU(2)$  is generally not manifest in a superpotential written in terms of  $\mathcal{N} = 1$  superfields, such as (2.1).

When the  $G_i$  are not simple, F and D-terms can be added to the Lagrangian as well,

$$\mathcal{L}_{F,D} = \sum_i \left[ \int d^4\theta d_i \text{Tr } V_i + \int d^2\theta f_i \text{Tr } \Phi_i + \text{c.c.} \right], \quad (2.2)$$

where  $d_i$  and  $f_i$  are (complex and real, respectively) parameters. The supersymmetric vacua of these theories are the solutions to the D and F-flatness conditions, which read

$$\begin{aligned} \sum_j a_{ij} q_i \left( A_{ij} A_{ij}^\dagger - B_{ij}^\dagger B_{ij} - A_{ji}^\dagger A_{ji} + B_{ij} B_{ij}^\dagger \right) + d_i &= 0, \\ \sum_j a_{ij} (A_{ij} B_{ji} - B_{ij} A_{ji}) + f_i &= 0, \\ \sum_j (A_{ij} \Phi_j - \Phi_i A_{ij}) &= 0, \\ \sum_j (B_{ij} \Phi_j - \Phi_i B_{ij}) &= 0. \end{aligned} \quad (2.3)$$

Consistency of the first equations with the adjacency of the quiver nodes will require that  $\sum_i f_i = \sum_i d_i = 0$ .

The one-loop exact beta-function for the gauge coupling  $g_i$  is

$$\beta(g_i) = -\frac{g_i^3}{16\pi^2} \frac{(2 + \frac{1}{2}\gamma_{\Phi_i}) T(G_i) - \frac{1}{2} \sum_j a_{ij} N_j (2 - \frac{1}{2}\gamma_{A_{ij}} - \frac{1}{2}\gamma_{B_{ji}})}{1 - \frac{g_i^2 T(G_i)}{8\pi^2}}. \quad (2.4)$$

Of particular interest to us are the class of  $\mathcal{N} = 2$  quiver gauge theories that can be obtained by studying D-branes at orbifold singularities [4]. By placing  $N$  D3-branes transverse to the orbifold  $\mathbb{C}^2/\Gamma$ , with  $\Gamma$  a finite subgroup of  $SU(2)$ , a 4D  $\mathcal{N} = 2$  gauge theory is obtained on the worldvolume. These  $\Gamma$  have an  $A$ – $D$ – $E$  classification. The  $U(N)$  gauge theory is broken to a product  $\prod_i U(N_i)$ , where  $N = \sum_i N_i$  and the  $N_i$  are in 1–1 correspondence with the vertices of the affine (or extended) Dynkin diagram. These vertices are the simple roots  $\alpha_i$ ,  $i = 1, \dots, n-1$  plus the extended root  $\alpha_0 = -\sum_{i=1}^{n-1} \alpha_i$  of the simply-laced  $A$ – $D$ – $E$  algebra. The inner product on the roots determines the adjacency matrix of the quiver diagram as  $a_{ij} = 2\delta_{ij} - \widehat{C}_{ij}$  where  $\widehat{C}_{ij}$  is the extended Cartan matrix. The hypermultiplets have charge  $q_i = 1$  and their moduli space describes the resolved ALE space  $\widetilde{\mathbb{C}^2/\Gamma}$ , while the scalars  $\Phi_i$  describe the remaining  $\mathbb{C}$  transverse directions.

The leading beta-function coefficient for  $g_i$  is

$$b_0^{(i)} = 2T(G_i) - \sum_j a_{ij}N_j = \sum_j \widehat{C}_{ij}N_j. \quad (2.5)$$

When only regular D3-branes are present,  $N_i = k_i \widetilde{N}$ , where the  $k_i$  are the Dynkin labels of the algebra. Since  $\sum_j \widehat{C}_{ij}k_j = 0$  as a Lie algebra identity, this field theory can be superconformal. If there are  $r_i$  fractional branes wrapping the  $i^{th}$  homology cycle of the ALE, then  $N_i = k_i \widetilde{N} + r_i$ . Now  $b_0^{(i)} = \sum_j \widehat{C}_{ij}r_j$  and the free field theory is not conformal. If  $b_0^{(i)} > 0$ , the (simple part of the) gauge group  $G_i$  is asymptotically free.

Let us examine the quiver gauge theory on regular D3-branes in some more detail. We set  $N_i = k_i \widetilde{N}$ . We apply the conditions for conformal invariance following [22]. Similar discussions for these theories appear in [5, 6, 10]. Then vanishing of the exact beta functions for the gauge couplings (2.4) lead to the conditions

$$k_i \gamma_{\Phi_i}(\tau_i, \lambda_i) + \frac{1}{2} \sum_j a_{ij} k_j (\gamma_{A_{ij}}(\tau_i, \lambda_i) + \gamma_{B_{ji}}(\tau_i, \lambda_i)) = 0, \quad (2.6)$$

while vanishing of the beta-functions for the  $\lambda_i$  require that

$$\begin{aligned} \gamma_{\Phi_i}(\tau_i, \lambda_i) + \sum_j a_{ij} (\gamma_{A_{ij}}(\tau_i, \lambda_i) + \gamma_{B_{ji}}(\tau_i, \lambda_i)) &= 0, \\ \gamma_{\Phi_i}(\tau_i, \lambda_i) + \sum_j a_{ij} (\gamma_{A_{ji}}(\tau_i, \lambda_i) + \gamma_{B_{ij}}(\tau_i, \lambda_i)) &= 0, \end{aligned} \quad (2.7)$$

where the  $\tau_i = \vartheta_i + 4\pi i/g_i^2$  are the complexified gauge couplings. The R-symmetry component  $SU(2)$  that acts on the hypermultiplets requires that  $\gamma_{B_{ji}} = \gamma_{A_{ij}}$ , while compatibility of (2.6) with the first of (2.7) requires that  $\gamma_{A_{ij}} \equiv \gamma_A$  are the same for all  $i, j$ . Furthermore, the two equations in (2.7) can be used to show that  $\gamma_{\Phi_i} = \gamma_{\Phi}$  for

all  $i$ . Then conformal invariance requires that

$$\gamma_\Phi(\tau_i, \lambda_i) + 2\gamma_A(\tau_i, \lambda_i) = 0. \quad (2.8)$$

This describes a fixed surface  $\lambda_i = \lambda_i(\tau_i)$ , which we will denote  $\mathcal{M}_\tau^{(n)}$ . Its structure is discussed in [23, 24].

We will now determine the anomalous dimensions by using the  $a$ -maximization technique of [14] to compute the exact  $\mathcal{N} = 1$  superconformal  $U(1)_R$  R-charges. Applications of this method to SQCD with adjoint matter were discussed in [25, 26], which contain important refinements. We use the relation between the anomalous dimension and R-charge,  $\gamma = \frac{3}{2}R - 1$ , to write

$$R_\Phi + 2R_A - 2 = 0. \quad (2.9)$$

Next we compute the  $a$ -charge, which is given by [16, 17]

$$a = \frac{3}{32} (3\text{Tr } R^3 - \text{Tr } R), \quad (2.10)$$

where the traces are performed over all fermions in the theory. For the quiver theories

$$a = \frac{3}{32} \left[ 2 \sum_i N_i^2 + \sum_i N_i^2 (3(R_{\Phi_i} - 1)^3 - (R_{\Phi_i} - 1)) \right. \\ \left. + \sum_{i < j} a_{ij} N_i N_j (3(R_{A_{ij}} - 1)^3 - (R_{A_{ij}} - 1)) \right. \\ \left. + 3(R_{B_{ji}} - 1)^3 - (R_{B_{ji}} - 1) \right]. \quad (2.11)$$

Maximizing this with respect to the constrained R-charges leads to

$$R_{\Phi_i} = R_{A_{ij}} = R_{B_{ji}} = \frac{2}{3}. \quad (2.12)$$

So  $a$ -maximization actually requires that the anomalous dimensions vanish, which is much stronger than the condition of conformal invariance (2.8). This occurs on the  $\mathcal{N} = 2$  fixed line  $\lambda_i = g_i$ . The result (2.12) agrees with that obtained by studying each gauge factor independently as a theory with  $N_c = N$ ,  $N_f = 2N$  and following [27].

### 3 Flux Deformation of IIB Orbifolds

The near horizon limit of the theory on  $N$  D3-branes at the orbifold  $\mathbb{C}^2/\Gamma$  is given by IIB supergravity in the  $\text{AdS}_5 \times S^5/\Gamma$  background [3]. Here since  $\Gamma \subset SU(2)$ ,



there is a singularity of  $S^5/\Gamma$  corresponding to a fixed circle. The isometry of  $S^5/\Gamma$  is  $SU(2) \times U(1)$ , which is the R-symmetry of the dual quiver gauge theory. The spectrum of this theory is discussed in [28, 29]. Among the states which are massless on  $\text{AdS}_5$  are  $n = \text{ord}(\Gamma)$  5D tensor multiplets.

One tensor multiplet comes from the untwisted sector of the orbifold theory and is dual to the chiral primary operator

$$\mathcal{O} = \text{Tr} \sum_i \phi_i^2 \quad (3.1)$$

and its descendants, where  $\phi_i$  is the scalar component of the superfield  $\Phi_i$ . There are 5 scalars in this tensor multiplet [30] and they have  $SU(2) \times U(1)$  charges  $\mathbf{1}_4$ ,  $\mathbf{3}_2$ , and  $\mathbf{1}_0$ . There is also a conjugate tensor multiplet with scalars  $(\mathbf{1}_{-4}, \mathbf{3}_{-2}, \mathbf{1}_0)$  corresponding to the antichiral operator,  $\text{Tr} \sum_i (\phi_i^\dagger)^2$ , conjugate to (3.1). These scalars are built from many types of 10D fields. The  $\mathbf{1}_{\pm 4}$  states are linear combinations of the lowest harmonics of metric ( $h^\alpha_\alpha$ ) and 4-form potential ( $a_{\alpha\beta\gamma\delta}$ ) degrees of freedom [31]. The  $\mathbf{3}_{\pm 2}$  arise from the lowest harmonic of the complex 2-form potential components ( $a_{\alpha\beta}$ ), while the pair  $2(\mathbf{1}_0)$  is the complex axion–dilaton.

The remaining  $n - 1$  tensor multiplets come from the twisted sectors and are dual to the differences

$$\mathcal{O}_i = \text{Tr} \left( \phi_i^2 - \phi_{i-1}^2 \right) \quad (3.2)$$

and descendants. Including the duals to the antichiral operators, the  $\mathbf{1}_{\pm 4}$ ,  $2(\mathbf{1}_0)$  scalars are linear combinations of harmonics of the periods of the complex 2-form potential over the compact 2-cycles of the  $\mathbb{C}^2/\Gamma$  orbifold. The  $\mathbf{3}_{\pm 2}$  states are the lowest harmonics of the moduli associated with varying the sizes of the compact 2-cycles, *i.e.*, they are the blow-up modes.

The operators (3.1), (3.2) are relevant, as are their level two descendants, which are built from untwisted and twisted sums of fermion bilinears  $\text{Tr} \chi_i \chi_i$ . Deformation of the conformal field theory by them will generate flows to an  $\mathcal{N} = 1$  conformal field theory [5, 6, 32, 24, 10, 7]. We will discuss the field theoretic properties of these fixed points in the next section. For now, we would like to discuss aspects of the dual geometry, following [5, 6, 10, 7].

The description of the RG flows generated by (3.1), (3.2) (and descendants) within the effective 5D  $\mathcal{N} = 4$   $SU(2) \times U(1)$  gauged supergravity was described in [7]. There it was found that the 5D dynamics is symmetric under an  $SU(n)$  acting on the tensor multiplets. In particular, this can be used to map a flow generated by a generic initial condition to one involving only the untwisted sector scalars.

The flows generated in the untwisted sector are completely analogous to those studied in the theory on  $S^5$ , without the orbifold, by [33]. In particular, the flows generated by the fermion bilinears, corresponding to the  $SU(2)$ –singlet component of

the  $\mathbf{3}_2$ , can be precisely mapped to the  $SU(2) \times U(1)$  preserving flow of [33]. The 5D flow solution of [33] was lifted to 10D in [12, 13]. The 10D solution involves a stretched and squashed metric on  $S^5$ , together with background fluxes for the complex 3-form and 5-form field strengths. In [7] it was argued that, since the 3 and 5-forms are invariant under  $\mathbb{Z}_n$ , the orbifold of the solution in [12, 13] is the 10D lift of the flow generated purely by the untwisted operator (3.1) in the corresponding quiver gauge theory.

The 10D geometries of the flows generated by purely twisted sector operators (3.2) are very different from that generated by the untwisted sector, however. Since the  $\mathbf{3}_2$  are blow-up modes, these flows correspond to desingularizing the orbifold singularity [5, 6]. Strictly speaking, the orbifold singularity is deformed. Introducing complex coordinates  $x$ ,  $y$ ,  $z$ , and  $t$ , if the ALE space is the  $A_{n-1}$  curve

$$xy = z^n \tag{3.3}$$

in  $\mathbb{C}^3$ , the effect of an single twisted sector operator is to deform this to

$$xy = z^{n-2}(z - \zeta t)(z + \zeta t), \tag{3.4}$$

which is an example of a generalized conifold. Part of the orbifold singularity at  $x = y = z = 0$  has been replaced with a conifold singularity at  $x = y = z = t = 0$ . The 2-sphere corresponding to the twisted sector of the operator now has area  $|\zeta t|$ . The ALE space is said to be fibered over the line parameterized by  $t$ . A generic twisted deformation leads to the curve

$$xy = \prod_i^n (z - \zeta_i t), \quad \sum_i \zeta_i = 0. \tag{3.5}$$

At the fixed point, the near horizon solutions involve only 5-form flux. Nevertheless, the 5D symmetry suggests that there should be some map between IIB fields corresponding to the different endpoints of the RG flows. However, metrics for the endpoints of these flows are only known in a small neighborhood of the conifold singularities [6].

When some untwisted sector flux is added, a 3-form flux is generated on the generalized conifold (3.5). This leads to a potential on the worldvolume of a probe brane in the generalized conifold geometry. We will find that the moduli space of the gauge theory on the probe brane is just the ALE fiber. This space corresponds to the minimum of the potential on the probe.

### 3.1 An illustration for $A_1$

The above picture is easiest to illustrate in the case of the  $\widehat{A}_1$  quiver theory. This theory has two adjoint scalars  $\Phi_1$ ,  $\Phi_2$  and there are two possible mass deformations.

The untwisted deformation (3.1) corresponds to adding the term

$$W_u = \Phi_1^2 + \Phi_2^2 \quad (3.6)$$

to the  $\mathcal{N} = 2$  superpotential. The geometry dual to the endpoint of the flow generated by (3.6) is a  $\mathbb{Z}_2$  orbifold of the solution of [12]. As remarked above, the metric is that of  $\text{AdS}_5$  times a stretched and squashed  $S^5/\mathbb{Z}_2$  and there is nonzero complex 3-form and 5-form flux. There is a  $\mathbb{Z}_2$  orbifold singularity which is a fixed line on the  $S^5/\mathbb{Z}_2$ . We will call this fixed point the PW point.

The twisted deformation (3.2) is the term

$$W_t = \Phi_1^2 - \Phi_2^2. \quad (3.7)$$

The gravity dual to the fixed point is now  $\text{AdS}_5$  times the base manifold  $T^{11}$  of the conifold [5]. There is 5-form flux, but no 3-form flux. The orbifold singularity on  $S^5$  is removed and  $T^{11}$  is smooth. We call the corresponding fixed point the KW point.

A general deformation

$$W = \frac{m_1}{2}\Phi_1^2 + \frac{m_2}{2}\Phi_2^2 \quad (3.8)$$

actually describes a point on the complex projective plane  $\mathbb{P}^1$ , which can also be identified with a 2-sphere. The reason is that the overall mass scale decouples in the IR, so the ratio  $m_2/m_1$  specifies the fixed point. The gravity dual for a general deformation (3.8) has not been constructed, but it involves both adding 3-form flux and desingularizing the orbifold singularity.

In terms of the homogeneous coordinates  $(m_1, m_2)$ , the PW point is  $(1, 1)$ , while  $(1, -1)$  is the KW point. In Figure 1, we represent the  $\mathbb{P}^1$  of fixed points with a 2-sphere and indicate the PW and KW points. We leave further discussion of the manifolds of fixed points to Section 5.

## 4 The Mass Deformation and $\mathcal{N} = 1$ Fixed Points

For simplicity, let us consider the superconformal quiver gauge theory on  $nN$  regular D3-branes at the  $\mathbb{C}^2/\mathbb{Z}_n$  singularity. This is the  $\widehat{A}_{n-1}$  theory. The gauge group is  $G = U(N)^n$  and the quiver corresponds to the affine  $\widehat{A}_{n-1}$  Dynkin diagram. The adjacency matrix is  $a_{ij} = \delta_{i-1,j} + \delta_{i+1,j}$ . We will use a slightly simplified notation for the hypermultiplets and define  $A_i \equiv A_{i,i+1}$ ,  $B_i \equiv B_{i+1,i}$ . The superpotential is

$$W_{\text{tree}} = \sum_i \lambda_i \text{Tr} \Phi_i (A_i B_i - B_{i-1} A_{i-1}). \quad (4.1)$$

As discussed in Section 3, deformations of this theory by relevant operators generate flows to other conformal fixed points. We will study the  $\mathcal{N} = 1$  superconformal field

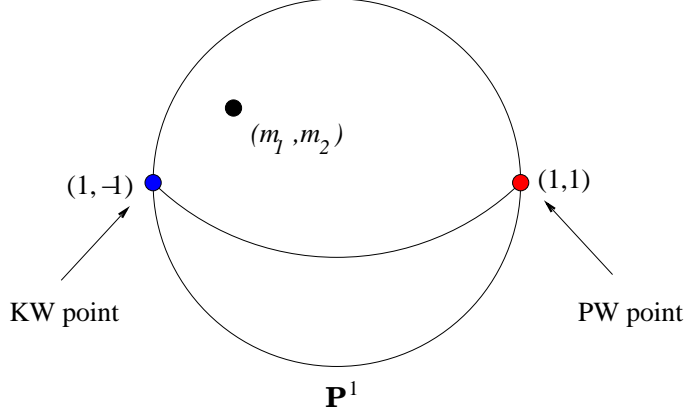


Figure 1: The  $\mathbb{P}^1$  of fixed points for the adjoint mass deformation (3.8) of the  $A_1$  quiver theory. The PW and KW points are indicated, as well as the generic point  $(m_1, m_2)$ .

theories obtained by a deformation by mass terms for the adjoint chiral fields

$$W_\Phi = \text{Tr} \sum_i \frac{m_i}{2} \Phi_i^2. \quad (4.2)$$

A purely twisted sector deformation will satisfy  $\sum_i m_i = 0$ , while  $\sum_i m_i \neq 0$  indicates that the untwisted sector deformation is present. The action on the hypermultiplet fields  $(A_i, B_i)$  of the  $SU(2)$  component of the  $\mathcal{N} = 2$  R-symmetry is preserved by (4.2) and becomes a global symmetry of the  $\mathcal{N} = 1$  theory.

For each field  $\Phi_i$  of mass  $m_i \neq 0$ , as we flow to scales  $\mu < m_i$ , we should integrate  $\Phi_i$  out of the dynamics. The remaining theory is that of the hypermultiplet fields and any massless  $\Phi_i$ , with the effective superpotential

$$W = \text{Tr} \left[ - \sum_{i|m_i \neq 0} \frac{h_i}{2} (A_i B_i - B_{i-1} A_{i-1})^2 + \sum_{i|m_i = 0} \lambda_i \Phi_i (A_i B_i - B_{i-1} A_{i-1}) \right], \quad (4.3)$$

where

$$h_i = \frac{\lambda_i^2}{m_i}. \quad (4.4)$$

When  $m_i \neq 0$ , we will use (4.4) to eliminate  $m_i$ .

Note that the presence of nonzero F-terms in (4.1) will result in the addition of a term

$$-\text{Tr} \sum_{i|m_i \neq 0} \frac{h_i f_i}{\lambda_i} (A_i B_i - B_{i-1} A_{i-1}) \quad (4.5)$$

to (4.3). This is a mass term and, were it to be nonzero, the corresponding  $(A_i, B_i)$  should be integrated out of the low-energy theory. Generally it is expected that this

theory will continue to RG flow into the IR and there will be no fixed point. If all of the  $m_i \neq 0$ , then for the mass term to be absent the  $f_i$  must satisfy

$$f_i = \frac{\lambda_i}{\lambda_{i-1}} \frac{h_{i-1}}{h_i} f_{i-1}. \quad (4.6)$$

For generic  $\lambda_i$  and  $m_i$ , this condition will fail to hold, so we conclude that the  $f_i = 0$ . However, we note that there is no such prohibition against adding D-terms to the theory.

Conformal invariance of (4.3) requires that

$$\left. \begin{aligned} \gamma_{\Phi_i}(\tau_i, \lambda_i, h_i) + \gamma_{A_i}(\tau_i, \lambda_i, h_i) + \gamma_{B_i}(\tau_i, \lambda_i, h_i) &= 0 \\ \gamma_{\Phi_i}(\tau_i, \lambda_i, h_i) + \gamma_{A_{i-1}}(\tau_i, \lambda_i, h_i) + \gamma_{B_{i-1}}(\tau_i, \lambda_i, h_i) &= 0 \end{aligned} \right\} \quad \forall i | h_i = 0, \quad (4.7)$$

$$\left. \begin{aligned} \gamma_{A_i}(\tau_i, \lambda_i, h_i) + \gamma_{B_i}(\tau_i, \lambda_i, h_i) + \frac{1}{2} &= 0 \\ \gamma_{A_{i-1}}(\tau_i, \lambda_i, h_i) + \gamma_{B_{i-1}}(\tau_i, \lambda_i, h_i) + \frac{1}{2} &= 0 \end{aligned} \right\}, \quad \forall i | h_i \neq 0.$$

One caveat about (4.7) is that whenever

$$h_i + h_{i-1} = 0 \quad (4.8)$$

for some  $i$ , then conformal invariance will require the weaker condition that

$$\gamma_{A_i} + \gamma_{B_i} + \gamma_{A_{i-1}} + \gamma_{B_{i-1}} + 1 = 0. \quad (4.9)$$

The  $SU(2)$  global symmetry requires that  $\gamma_{A_i} = \gamma_{B_i} \equiv \gamma_i$ . Let us first consider the case that all  $m_i \neq 0$  and that (4.8) does not occur. Then the superpotential (4.3) depends only on  $h_i$ . The (4.7) require that

$$\gamma_i(\tau_i, h_i) + \frac{1}{4} = 0. \quad (4.10)$$

This is  $n$  equations in  $2n$  unknowns. There is a manifold of fixed points [6, 7]. Whenever at least one  $m_i$  is finite, the  $h_i(\tau_i)$  form a  $\mathbb{P}^{n-1}$  manifold of fixed points<sup>1</sup>, with inhomogeneous coordinates  $h_i/h_j$ .

A special point is obtained when all  $m_i \rightarrow \infty$ . In this limit the superpotential (4.3) vanishes. Each gauge theory is a copy of SQCD with  $N_f = 2N_c$  flavors, so it is in the conformal window of [34]. We therefore expect that this theory also flows to a superconformal fixed point.

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<sup>1</sup>The geometry is  $\mathbb{P}^{n-1}$  for both the bare  $h_i$  and the renormalized couplings  $h_i(\tau_i)$ , since (4.10) ensures a common wavefunction renormalization for all of the quartic operators in (4.3). Therefore the ratios  $h_i/h_j$  are not renormalized.

In the case that (4.8) holds, then some of the equations (4.10) will be replaced by

$$\gamma_i(\tau_i, h_i) + \gamma_{i-1}(\tau_i, h_i) + \frac{1}{2} = 0. \quad (4.11)$$

We still have  $n$  equations in  $2n$  unknowns and the same picture of a  $\mathbb{P}^{n-1}$  manifold of fixed points emerges.

When some  $m_i = 0$ , we can apply cyclicity of the quiver to (4.7) to find that

$$\gamma_{\Phi_i}(\tau_i, \lambda_i, h_i) = \frac{1}{2}, \quad m_i = 0. \quad (4.12)$$

These equations can be used to find the  $\lambda_i(\tau_i, h_i)$ , which describe a manifold of the form  $\mathcal{M}_\tau^{(\nu)}$ , where  $\nu$  counts the number of the  $m_i = 0$ . Then the equations (4.10) or (4.11) will determine the  $h_i(\tau_i)$ , which now parameterize a  $\mathbb{P}^{n-\nu-1}$ . This  $\mathbb{P}^{n-\nu-1} \subset \mathbb{P}^{n-1}$  as the vanishing loci of sets of  $\nu$  of the  $h_i^{-1}$  in the theory with  $m_i \neq 0$  [7].

#### 4.1 Analysis by $a$ -maximization

We would like to analyze the fixed manifolds defined by (4.7) in more detail. We will again determine the exact R-charges by  $a$ -maximization [14]<sup>2</sup>. The main difference with the above approach is that we do not need to use the  $SU(2)$  symmetry. Nevertheless, we will still recover the result (4.10).

We can rewrite (4.7)

$$\left. \begin{aligned} R_{\Phi_i} + R_{A_i} + R_{B_i} - 2 &= 0 \\ R_{\Phi_i} + R_{A_{i-1}} + R_{B_{i-1}} - 2 &= 0 \end{aligned} \right\} \quad \forall i | m_i = 0, \quad (4.13)$$

$$\left. \begin{aligned} R_{A_i} + R_{B_i} - 1 &= 0 \\ R_{A_{i-1}} + R_{B_{i-1}} - 1 &= 0 \end{aligned} \right\} \quad \forall i | m_i \neq 0.$$

Next we compute the  $a$ -charge, which is given by [16, 17]

$$a = \frac{3}{32} (3 \text{Tr } R^3 - \text{Tr } R), \quad (4.14)$$

where the traces are performed over all fermions in the theory. Because some adjoints are integrated out, we need to modify the formula (2.11) accordingly. For the  $\widehat{A}_{n-1}$  theories we find that

$$a = \frac{3}{32} \left[ 2nN^2 + N^2 \sum_{i|m_i=0} (3(R_{\Phi_i} - 1)^3 - (R_{\Phi_i} - 1)) \right. \\ \left. + N^2 \sum_i (3(R_{A_i} - 1)^3 - (R_{A_i} - 1) + 3(R_{B_i} - 1)^3 - (R_{B_i} - 1)) \right]. \quad (4.15)$$

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<sup>2</sup>A related discussion appears in [35, 36]

Let us first examine  $\widehat{A}_1$  with  $m_1, m_2 \neq 0$ . Then both  $\Phi_{1,2}$  are integrated out and (4.13) become

$$R_{A_1} + R_{B_1} - 1 = R_{A_2} + R_{B_2} - 1 = 0. \quad (4.16)$$

The  $a$ -charge (4.15) is

$$a = \frac{3}{32} [2nN^2 + N^2 (3 [(R_{A_1} - 1)^3 - R_{A_1}^3] + 3 [(R_{A_2} - 1)^3 - R_{A_2}^3] + 2)]. \quad (4.17)$$

The only extremum is a maximum at the point

$$R_{A_1} = R_{A_2} = R_{B_1} = R_{B_2} = \frac{1}{2}. \quad (4.18)$$

The extremization equations for general  $\widehat{A}_{n-1}$  with all  $m_i \neq 0$  will take the same form as the ones derived from (4.17). Therefore the R-charges at generic fixed points when all  $m_i \neq 0$  always satisfy

$$R_{A_i} = R_{B_i} = \frac{1}{2}, \quad R_{\Phi_i} = 1. \quad (4.19)$$

This computation justifies the R-charge assignments of [32]. In fact, since (4.13) imply that

$$R_{A_i} + R_{B_i} = R_{A_{i+1}} + R_{B_{i+1}}, \quad m_i = 0, \quad (4.20)$$

the cyclicity of the quiver will result in the R-charge assignments (4.19) even when  $\mu < n$  of the  $m_i = 0$ . Since the fermionic components of  $\Phi_i$  drop out of the R-charge traces, the computation of [32] that  $\frac{c_{\text{IR}}}{c_{\text{UV}}} = \frac{27}{32}$  is valid for all flows away from  $\widehat{A}_{n-1}$  generated by masses (4.2). This fits well with the supergravity analysis of [7].

We note that all of the gauge invariant operators at the fixed point satisfy the unitarity bound  $R \geq \frac{2}{3}$  and that  $a_{\text{UV}} > a_{\text{IR}} = \frac{27}{32}a_{\text{UV}}$  so that the  $a$ -theorem is satisfied for these flows [10].

## 4.2 Moduli space geometry

We would now like to study the moduli space of the theory (4.3). The F-flatness conditions for arbitrary  $m_i$  are

$$\begin{aligned} & \delta_{m_i,0} \lambda_i (A_i B_i - B_{i-1} A_{i-1}) = 0, \\ & -\delta_{m_i,0} \lambda_i B_i \Phi_i + \delta_{m_{i-1},0} \lambda_{i+1} \Phi_{i+1} B_i \\ & + (1 - \delta_{m_i,0}) h_i (B_i A_i B_i - B_i B_{i-1} A_{i-1}) \\ & + (1 - \delta_{m_{i+1},0}) h_{i+1} (B_i A_i B_i - A_{i+1} B_{i+1} B_i) = 0. \end{aligned} \quad (4.21)$$

There is an additional equation arising from  $\partial W/\partial B_i$  which is analogous to the second equation above.

The general features of the geometry of the moduli space can be obtained by studying the  $U(1)_{\text{diag.}} \subset U(N)^n$  degrees of freedom

$$A_i = a_i \mathbb{1}_{N \times N} + \cdots, \quad B_i = b_i \mathbb{1}_{N \times N} + \cdots, \quad \Phi_i = \phi_i \mathbb{1}_{N \times N} + \cdots. \quad (4.22)$$

The F-flatness conditions will result in a moduli space for these degrees of freedom which we can denote by  $\mathcal{M}$ . Then projecting the F-flatness conditions onto the other components of the Cartan subalgebra of  $U(N)$  will lead to additional copies of  $\mathcal{M}$ . The Weyl group  $S_N$  acts on these components, so the full moduli space is the symmetric product

$$S^N \mathcal{M} = (\mathcal{M})^N / S_N. \quad (4.23)$$

In the following, we will refer to  $\mathcal{M}$  as the “moduli space.”

We define  $\mathbb{Z}_n$ -invariant coordinates by

$$x = \prod_i a_i, \quad y = \prod_i b_i, \quad z_i = a_i b_i. \quad (4.24)$$

Then (4.21) become

$$\begin{aligned} \delta_{m_i,0}(z_i - z_{i-1}) &= 0, \\ -\delta_{m_i,0}\phi_i + \delta_{m_{i+1},0}\phi_{i+1} + (1 - \delta_{m_i,0})h_i(z_i - z_{i-1}) \\ &\quad + (1 - \delta_{m_{i+1},0})h_{i+1}(z_i - z_{i+1}) = 0. \end{aligned} \quad (4.25)$$

We first consider the case that all  $m_i \neq 0$ . Then, using (4.4), the equations (4.25) become a matrix equation

$$M_{ij}z_j = 0, \quad M_{ij} = (h_i + h_{i+1})\delta_{ij} - h_{i+1}\delta_{i+1,j} - h_i\delta_{i-1,j}. \quad (4.26)$$

We would like to compute the rank of  $M$ . We find that we can express

$$\det M = (h_1 + h_2 - h_1 - h_2) \left( \prod_{i=1}^n h_i \right) \left( \sum_{i=1}^n h_i^{-1} \right) = 0. \quad (4.27)$$

Since  $\det M = 0$ ,  $M \cdot z = 0$  always has non-trivial solutions. We further note that if  $S$  is any  $n-1 \times n-1$  submatrix of  $M$ , then

$$\det S = f(h_i) \left( \sum_{i=1}^n h_i^{-1} \right). \quad (4.28)$$

Therefore  $M$  has rank  $n-1$  when  $\sum_i h_i^{-1} \neq 0$  and has rank  $n-2$  when  $\sum_i h_i^{-1} = 0$ .



When  $M$  has rank  $n - 1$ , the solutions of (4.26) are given by  $z_i = c(z)$  for some function  $c(z) \neq 0$ . From the homogeneity of the problem, we should take  $c(z)$  to be linear,  $c(z) = z$ , where we have absorbed a possible numerical factor into the definition of  $z$ . Then we find that the moduli space is given by

$$xy = z^n, \quad (4.29)$$

*i.e.*, it is the singular  $A_{n-1}$  curve in  $\mathbb{C}^3$ .

Recall that the existence of the fixed point requires that the F-term coefficients  $f_i = 0$ . However, we are free to add D-terms, so the singularity of (4.29) can be resolved, but not deformed by a modification of the complex structure. This has an interpretation in the string dual picture. Evidently the untwisted 3-form flux not only generates a potential on the probe brane, but it also creates an obstruction to complex deformation of the ALE space. This is possible, since the metric on the ALE space is no longer Ricci flat. Therefore there is no hyperKähler isometry to relate the resolutions to the complex structure deformations.

When  $M$  has rank  $n - 2$ , we again have the solution  $z_i = z$ . However now there are also solutions to the submatrix equation  $S \cdot z = 0$ . By operations on the rows of  $M$ , these generate additional solutions to  $M \cdot z = 0$  of the form  $z_i = \gamma_i \tau(t)$ , where  $t$  is an independent complex variable. However, these  $z_i$  should scale as  $t \sim \langle \text{Tr } \Phi_i \rangle$ , so it is natural to choose  $\tau(t) = t$ . Putting these solutions together, we have  $z_i = z - \gamma_i t$ . By a translation in  $z$ , we can set  $\sum_i \gamma_i = 0$ . Note that  $M \cdot \gamma = 0$  and  $\sum_i \gamma_i = 0$  are  $n - 1$  independent equations for the  $n$   $\gamma_i$ . These have the solution  $\gamma_i - \gamma_{i-1} = h_i^{-1}$ . The remaining degree of freedom can be absorbed into a rescaling of  $t$ .

The moduli space is

$$\begin{aligned} xy &= \prod_{i=1}^n (z - \gamma_i t), \\ \gamma_i &= \sum_{j=1}^i h_j^{-1} - \frac{1}{n} \sum_{j=1}^n (n - j) h_j^{-1}. \end{aligned} \quad (4.30)$$

The moduli space is a deformed  $A_{n-1}$  curve fibered over a complex line parameterized by  $t$ . This manifold is a generalized conifold, and its appearance in this field theory was discussed by [6, 9]. The parameters  $h_i$  determine the complex structure moduli of this manifold.

Now suppose that  $\mu$  of the  $m_i$  vanish. The equations (4.25) now give an equation of the form

$$\widetilde{M}_{ij} Z_j = 0, \quad Z_i = (z_1, \dots, z_{n-\mu}, \phi_1, \dots, \phi_\mu), \quad (4.31)$$

where we have made a relabeling of variables. The matrix  $\widetilde{M}$  has  $\det \widetilde{M} = 0$ . Also the submatrix acting on the  $z_i$  subspace has the same form as  $M_{ij}$  in (4.26). We find that

$\widetilde{M}$  has rank  $n - \mu - 1$  if  $\sum_i h_i^{-1} \neq 0$  and rank  $n - \mu - 2$  if  $\sum_i h_i^{-1} = 0$ . In the former case, we obtain the resolvable  $A_{n-1}$  curve (4.29) as the moduli space. For the latter, we will find that

$$xy = z^\mu \prod_{i=1}^{n-\mu} (z - \gamma_i t). \quad (4.32)$$

These are generalized conifolds corresponding to partial resolutions of the  $A_{n-1}$  singularities. Existence of the fixed point rules out adding F-terms, except perhaps in some very special situations.

## 5 The Spectrum of Marginal Operators at the Fixed Points

We would like to further analyze the manifolds of fixed points discussed in Section 4. Specifically we would like elucidate the form of the marginal operators that parameterize the manifolds of fixed points.

Motion in the moduli space  $\mathcal{M}_\tau^{(\nu)}$  of gauge couplings is generated by exactly marginal operators corresponding to the differences between the gauge kinetic energies. We are interested in motion on the moduli space of mass deformations, parameterized by the  $h_i(\tau_i)$ , so we will study the theories with all  $m_i \neq 0$ . These fixed points are defined by superpotentials

$$W = -\text{Tr} \sum_i \frac{h_i}{2} T_i, \quad (5.1)$$

$$T_i = (A_i B_i - B_{i-1} A_{i-1})^2.$$

Let us first consider the  $A_1$  theory. The moduli space is a  $\mathbb{P}^1$  parameterized by homogeneous coordinates  $(h_1, h_2)$ , as depicted in Figure 1. We find that we can rewrite (5.1) as

$$W = -\frac{1}{4} \text{Tr} \left[ (h_1 - h_2) (A_1 B_1 B_2 A_2 - A_2 B_2 B_1 A_1) \right. \\ \left. - (h_1 + h_2) \left( (A_1 B_1 - B_2 A_2)^2 + (A_2 B_2 - B_1 A_1)^2 \right) \right]. \quad (5.2)$$

The case  $m_1 + m_2 = 0$  was studied in [5]. They argued that a  $\mathbb{Z}_2$  symmetry could be used to fix the gauge couplings to be equal along the flow to the fixed point,  $\tau_1 = \tau_2$ . Therefore we also have  $h_1 + h_2 = 0$ . The superpotential (5.2) reduces to

$$W = \lambda_{\text{KW}} W_{\text{KW}} = \lambda_{\text{KW}} \text{Tr} (A_1 B_1 B_2 A_2 - A_2 B_2 B_1 A_1), \quad (5.3)$$

which agrees with their superpotential after one notes the change of notation  $A_2^{\text{here}} = B_2^{\text{KW}}$ ,  $B_2^{\text{here}} = A_2^{\text{KW}}$ . Note that  $m_1 + m_2 = 0$  corresponds to the point  $(1, -1)$  on  $\mathbb{P}^1$ ,

which we referred to as the KW point in subsection 3.1. It was noted in [5] that this point has an  $SU(2) \times SU(2)$  global symmetry. In our notation, each factor acts independently on the doublets  $(A_1, B_2)$  and  $(A_2, B_1)$ . The  $SU(2)$  at a generic point of  $\mathbb{P}^1$  which was inherited from the  $\mathcal{N} = 2$  R-symmetry is the diagonal subgroup of this enhanced symmetry. Furthermore, [5] explained how this theory described  $N$  D3-branes on the conifold. The result (4.30) reduces appropriately.

The point on  $\mathbb{P}^1$  which is antipodal to the KW point is  $m_1 = m_2$ , which is the PW point. This point also has a  $\mathbb{Z}_2$  symmetry which can be used to set  $\tau_1 = \tau_2$ , so  $h_1 = h_2$  as well. Then (5.2) reduces to

$$W = \lambda_{\text{PW}} W_{\text{PW}} = \lambda_{\text{PW}} \text{Tr} \left[ (A_1 B_1 - B_2 A_2)^2 + (A_2 B_2 - B_1 A_1)^2 \right]. \quad (5.4)$$

In [7] it was argued that this point describes  $N$  D3-branes at a cone over a  $\mathbb{Z}_2$  orbifold of the fixed point solution of [12, 13]. This solution has non-zero complex 3-form flux, which generates a potential for a D3-brane probe [15]. The minimum of this potential determines the moduli space to be a singular  $A_1$  ALE space, in agreement with the analysis leading to (4.29).

Now the whole  $\mathbb{P}^1$  of fixed point theories can be described by rewriting (5.2) as

$$W = \lambda_{\text{KW}} W_{\text{KW}} + \lambda_{\text{PW}} W_{\text{PW}}. \quad (5.5)$$

The  $\mathbb{P}^1$  is recovered as the fixed line of solutions to the equations

$$\begin{aligned} \gamma_{W_{\text{KW}}}(\tau, \lambda_{\text{KW}}, \lambda_{\text{PW}}) + 2 &= 0, \\ \gamma_{W_{\text{PW}}}(\tau, \lambda_{\text{KW}}, \lambda_{\text{PW}}) + 2 &= 0. \end{aligned} \quad (5.6)$$

We can take the difference

$$\mathcal{O} = \frac{1}{2} (W_{\text{KW}} - W_{\text{PW}}) = 2(A_2 B_2 - B_1 A_1)^2 \quad (5.7)$$

to be the exactly marginal operator which generates translations on the  $\mathbb{P}^1$  manifold of fixed points.

In the  $A_{n-1}$  case, we will have a  $\mathbb{P}^{n-1}$  manifold of fixed points specified by the conditions

$$\gamma_i(\tau_i, h_i) + 2 = 0 \quad (5.8)$$

on the anomalous dimensions of the operators  $T_i$  defined in (5.1). The analog of the PW point is the point  $m_i = m$ ,  $\forall i$ . Now this point describes  $N$  D3-branes at a cone over a  $\mathbb{Z}_n$  orbifold of the fixed point solution of [12, 13]. The  $\mathbb{Z}_n$  symmetry can be used to set the gauge couplings equal  $\tau_i = \tau$ , so that  $h_i = h$ ,  $\forall i$ . The operator defining this point is

$$W_{\text{PW}} = \text{Tr} \sum_i T_i. \quad (5.9)$$

The equation  $\sum_i h_i^{-1} = 0$  defines a subvariety of  $\mathbb{P}^{n-1}$  that is isomorphic to  $\mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$ . Therefore the analog of the KW fixed point is a  $\mathbb{P}^{n-2}$  submanifold. We will refer to this submanifold as the conifold subspace. Let us consider the perturbation of the superpotential at a point on the conifold subspace by the marginal operator

$$\mathcal{O} = \text{Tr} \sum_i c_i T_i. \quad (5.10)$$

This perturbation takes us to a new point defined by the superpotential

$$W' = \text{Tr} \sum_i (h_i + c_i) T_i \equiv \text{Tr} \sum_i h'_i T_i. \quad (5.11)$$

In general the point specified by the  $h'_i$  is no longer on the hyperplane  $\sum_i h_i^{-1} = 0$ . The condition that this point is still on the conifold subspace is

$$0 = \sum_i (h'_i)^{-1} = \sum_i \frac{1}{h_i + c_i}. \quad (5.12)$$

This has the solution

$$c_i = -\frac{h_i \gamma_i}{1 + h_i \gamma_i}, \quad \sum_i \gamma_i = 0. \quad (5.13)$$

Then (5.10) takes the point  $h_i^{-1}$  to the point  $(h'_i)^{-1} = h_i^{-1} + \gamma_i$ .

A basis for general perturbations of a fixed point can be chosen as the operators

$$\mathcal{O}_i = \text{Tr} (T_i - T_{i-1}). \quad (5.14)$$

As we found above, a generic perturbation (5.10) will move a point on the conifold subspace off into the  $\mathbb{P}^{n-1}$ . Perturbations which move a point to another point on the conifold subspace depend on the initial condition according to (5.13). Therefore a basis for perturbations within the conifold subspace is

$$\mathcal{O}_i(\gamma) = \text{Tr} \left[ -\frac{h_i \gamma}{1 + h_i \gamma} T_i + \frac{h_{i-1} \gamma}{1 - h_{i-1} \gamma} T_{i-1} \right]. \quad (5.15)$$

These take  $(h_1^{-1}, \dots, h_{i-1}^{-1}, h_i^{-1}, \dots, h_n^{-1})$  to  $(h_1^{-1}, \dots, h_{i-1}^{-1} - \gamma, h_i^{-1} + \gamma, \dots, h_n^{-1})$ . The operators (5.15) form an Abelian group under compositions  $\mathcal{O}_i(\gamma) \cdot \mathcal{O}_{i'}(\gamma')$ .

## 6 Deformation of $\mathcal{N} = 2$ Conformal Theories by General $W(\Phi_i)$

One can also consider deformations by general polynomials

$$\begin{aligned} W(\Phi_1, \dots, \Phi_n) &= \sum_i W_i(\Phi_i), \\ W_i(\Phi_i) &= \text{Tr} \sum_{r=0}^k \frac{g_{r+1}^{(i)}}{r+1} \Phi_i^{r+1}, \end{aligned} \quad (6.1)$$

as in [9, 10, 37]. For a generic perturbation, it is not possible to analyze the theory by integrating out the  $\Phi_i$ , so one must study the F-flatness conditions (we specialize to  $\widehat{A}_{n-1}$  for convenience)

$$\begin{aligned}\lambda_i (A_i B_i - B_{i-1} A_{i-1}) + W'_i(\Phi_i) &= 0, \\ -\lambda_i B_i \Phi_i + \lambda_{i+1} \Phi_{i+1} B_i &= 0, \\ \lambda_i \Phi_i A_i - \lambda_{i+1} A_i \Phi_{i+1} &= 0.\end{aligned}\tag{6.2}$$

The first equation is consistent with cyclicity for two cases. Either  $\sum_i W'_i(\Phi_i)/\lambda_i = 0$  or  $\langle \Phi_i \rangle = 0$  for all  $i$ . In the latter case, we must also demand that  $\sum_i g_1^{(i)} = 0$ , which is just the familiar condition on the F-terms. When  $\sum_i W'_i(\Phi_i)/\lambda_i \neq 0$ , the moduli space is an ALE space.

When  $\sum_i W'_i(\Phi_i)/\lambda_i = 0$ , the second and third equations require that  $\lambda_i \Phi_i = \Phi$  for all  $i$ .

Then the equations for the  $U(1)_{\text{diag}}$  degrees of freedom (4.22) become

$$z_i - z_{i+1} + \frac{1}{\lambda_i} W'_i(t) = 0.\tag{6.3}$$

We have absorbed the gauge coupling into the definition of the  $\mathbb{Z}_n$ -invariant coordinates (4.24) and introduced the coordinate  $t = \frac{1}{N} \text{Tr } \Phi$ . The equations (6.3) can be solved by setting  $z_1 = z + c$  and computing the other  $z_i$  by recursion. The moduli space obtained is [9, 37]

$$\begin{aligned}xy &= \prod_i (z - \tau_i(t)), \\ \tau_i(t) &= \sum_{j=1}^i \frac{W'_j(t)}{\lambda_j} - \frac{1}{n} \sum_{j=1}^n (n-j) \frac{W'_j(t)}{\lambda_j}.\end{aligned}\tag{6.4}$$

The  $\tau_i(t)$  are degree  $n-1$  polynomials in  $t$ . This is the most general deformation of the  $A_{n-1}$  curve to a generalized conifold [11, 9].

## 6.1 Restrictions from conformal invariance

This computation of the moduli space is not the complete story. After deforming the  $\mathcal{N} = 2$  fixed point by (6.1), the theory should flow to some conformal fixed point in the IR. The moduli space should then reflect the geometry dual to this IR fixed point. However, (6.1) generally contains operators which are irrelevant at the  $\mathcal{N} = 2$  fixed point.

This issue was addressed in [10] by an argument that the operators in (6.1) are actually dangerously irrelevant. In analogy to [38], they argue that at large  $g_{k+1}^{(i)}$  there is a fixed point where the operators

$$W_i(\Phi_i) = \text{Tr} \frac{g_{k+1}^{(i)}}{k+1} \Phi_i^{k+1} \quad (6.5)$$

become marginal. However, we will now present an argument that there are no dangerously irrelevant operators for  $k > 2$ .

Assuming that some version of an  $a$ -theorem for 4D RG flows is true, a crucial criterion for the existence of a fixed point is that the  $a$ -theorem is satisfied for flows generated by relevant deformations at the candidate fixed point [26]. In [10] the  $a$ -charge for the candidate fixed points generated by the perturbations (6.5) was found to be

$$a_k = \frac{27k^2 \tilde{N}^2 |\Gamma|}{16(k+1)^3}. \quad (6.6)$$

A class of relevant operators at the  $\text{Tr} \Phi^{k+1}$  candidate fixed points are the operators  $\text{Tr} \Phi^{k'+1}$  with  $k' < k$ . Perturbations by  $\text{Tr} \Phi^{k'+1}$  would drive the theory toward the candidate fixed point where  $\text{Tr} \Phi^{k'+1}$  becomes marginal. However, since

$$\frac{da_k}{dk} = -(27\tilde{N}^2 |\Gamma|) \frac{k(k-2)}{16(k+1)^4} \quad (6.7)$$

is negative definite for  $k > 2$ , the  $a_k$  charge for these candidate fixed points is a strictly decreasing function. Therefore  $a_{k'} > a_k$  whenever  $k' < k$  and these flows would always violate the  $a$ -theorem. We conclude that  $\text{Tr} \Phi^{k+1}$ ,  $k > 2$  do *not* generate new superconformal fixed points. The operators  $\text{Tr} \Phi^{k+1}$  are simply irrelevant when  $k > 2$ .

This result does not contradict the fact that the operators analogous to (6.1) in SQCD with one adjoint are dangerously irrelevant [38, 25]. If we set all but one gauge coupling to zero, we obtain a  $U(N_c)$  gauge theory with  $N_f = 2N_c$  quarks. We can apply the analysis of [25] to this theory. They found that the operators  $\text{Tr} \Phi^{k+1}$  defined new fixed points for  $N_c$  and  $N_f$  satisfying  $N_c/N_f > x_k$ , where for small  $k$ ,

$$x_k = \sqrt{\frac{1}{20} \left( \frac{(5k-4)^2}{9} + 1 \right)}. \quad (6.8)$$

As  $x_k > \frac{1}{2}$  for  $k > 2$ , these fixed points never exist for the value  $N_f = 2N_c$  corresponding to the quiver theories.

As noted in [10], the candidate fixed point for the cubic operator  $\text{Tr} \Phi^3$  satisfies  $a_{k=2} = a_{\text{free}}$ . Since  $a_{k=1} < a_{k=2}$ , there is no  $a$ -theorem violation for the deformation of this by a mass term  $\text{Tr} \Phi^2$ . In adjoint SQCD, [25] determined that  $x_2 = \frac{1}{2}$ , so it

is possible that the cubic operator in this theory is marginally relevant. However, the present analysis cannot decisively rule out or prove the existence of these candidate fixed points. We note that a perturbative analysis described in [10] suggests that  $\Phi_i^3$  is marginally irrelevant for small couplings, but the large coupling behavior is still unknown.

## 6.2 The IR fixed point moduli spaces revisited

The above results imply that if we add a potential  $W(\Phi_i)$  to the  $\mathcal{N} = 2$  conformal theory, the higher order terms in  $W(\Phi_i)$  are irrelevant. The analysis that lead to (6.4) is strictly only valid in the UV, where the irrelevant operators in  $W(\Phi_i)$  are still important. At the fixed point in the IR, the effective  $W(\Phi_i)$  will only contain marginal operators, which can come from operators that were relevant in the UV. If  $W(\Phi_i)$  does not contain any relevant operators, it is possible that some are generated at critical points of  $W(\Phi_i)$ . If they are not, the theory will flow back to the undeformed  $\mathcal{N} = 2$  theory, with its orbifold moduli space.

We will consider the case that the  $\Phi_i$  are near a critical point of  $W(\Phi_i)$  and compute the relevant part of  $W(\Phi_i)$  at this critical point. This will be a sum of mass terms for the perturbations around the critical point and we can compute the moduli spaces reliably using the analysis of Section 4. We will then argue that this is consistent with the validity of (6.4) away from the critical points. However, it will still be important that  $W(\Phi_i)$  contain mass terms in the UV in order to remove the orbifold singularities in the moduli space.

Suppose that  $\phi_i$  is a critical point of  $W_i(\Phi_i)$ . Applying the F-flatness conditions (6.2), we find that  $\phi_i = \phi, \forall i$ . Clearly this is easiest to accomplish if  $W_i(\phi)/\lambda_i = W(\phi), \forall i$ , but we will not require this. We now set  $\Phi_i = \phi + \tilde{\Phi}_i/\lambda_i$  and expand to quadratic order

$$W_i(\Phi_i) = W_i(\phi) + \frac{1}{2}W_i''(\phi) \left( \frac{\tilde{\Phi}_i}{\lambda_i} \right)^2 + \dots \quad (6.9)$$

The F-flatness conditions are now

$$\begin{aligned} (A_i B_i - B_{i-1} A_{i-1}) + \frac{W_i''(\phi)}{\lambda_i^2} \tilde{\Phi}_i &= 0, \\ -B_i \tilde{\Phi}_i + \tilde{\Phi}_{i+1} B_i &= 0, \\ \tilde{\Phi}_i A_i - A_i \tilde{\Phi}_{i+1} &= 0. \end{aligned} \quad (6.10)$$

If  $\sum_i W_i''(\phi)/\lambda_i^2 \neq 0$ , then the first equation of (6.10) is consistent with cyclicity only for  $\tilde{\Phi}_i = 0$ . We will recover a moduli space which is just the ALE space.

If  $\sum_i W_i''(\phi)/\lambda_i^2 = 0$ , then the second and third equations of (6.10) require that  $\lambda_i \tilde{\Phi}_i = \tilde{\Phi}$ ,  $\forall i$ . Now the solution presented in Section 4 involves a coordinate  $t$  that is related to the VEVs of  $\langle \text{Tr } \tilde{\Phi} \rangle$ . Because we are close to the critical point, we set  $t \equiv \phi + \tilde{t}$ , where  $\tilde{t} \equiv \frac{1}{N} \langle \text{Tr } \tilde{\Phi}_i \rangle$ . The moduli space is then

$$xy = \prod_i (z - \tau_i(\tilde{t}, \phi)),$$

$$\tau_i(\tilde{t}, \phi) = \tilde{t} \left[ \sum_{j=1}^i \frac{W_j''(\phi)}{\lambda_j^2} - \frac{1}{n} \sum_{j=1}^n (n-j) \frac{W_j''(\phi)}{\lambda_j^2} \right]. \quad (6.11)$$

The result (6.11) agrees with the expansion of (6.4) around the critical points,  $t = \phi + \tilde{t}$ , of the  $W_i(t)$ . It is natural then to assume that (6.4) is the correct result away from the critical points. Evidently, when the irrelevant  $\Phi_i^{k+1}$  decouple as the theory flows to the IR, the data of their coupling constants is reflected in the running of the couplings of the less irrelevant operators. From (6.4) we conclude that the theory flows to a fixed point generated by an effective mass  $W_i'(t)$  for the  $\Phi_i$ .

In order for (6.4) not to have orbifold singularities, it is crucial that  $W(\Phi_i)$  actually contains non-zero bare mass terms for all of the  $\Phi_i$ . Suppose that  $W_i(\Phi_i)$  does not have a mass term. Then the  $i^{\text{th}}$  sector will flow to the least irrelevant monomial  $W_i(\Phi_i) \sim \Phi_i^{k+1}$ . This operator is still irrelevant if  $k > 2$ , but it is possible that relevant operators are generated at a critical point. However, the only critical point for this potential is  $\phi_i = 0$ , so  $W_i''(\phi_i) = 0$  (no relevant operator is generated). Then at least one of the differences  $\tau_i - \tau_j$  in (6.11) will vanish. Correspondingly, (6.11) only corresponds to a partial resolution of the  $A_{n-1}$  orbifold singularity. In some cases, it may be possible to add F-terms to generate a critical point for  $W_i(\Phi_i)$  at  $\phi_i = \phi \neq 0$ .

## 7 Nonconformal Theories and Quantum Moduli Spaces

A generic  $\mathcal{N} = 2$  quiver gauge theory will not be conformal. Nonconformal theories can be obtained from the  $\mathcal{N} = 2$  theory on D3-branes at an orbifold by adding fractional branes. Deformations of these theories by adjoint masses lead to  $\mathcal{N} = 1$  field theories that illustrate many interesting features [18–21, 9, 10].

There are two effects we want to analyze. Firstly, since the coupling constants in these theories are running, it is common that some of the gauge groups will become strongly coupled in the IR. At these points, a better description of the theory is in terms of a Seiberg dual theory [27]. As one continues to flow into the IR, the theory can undergo repeated Seiberg dualities, leading to a duality cascade [21]. We therefore analyze the effect of performing a Seiberg duality at a node of the quiver.



Secondly, these  $\mathcal{N} = 1$  theories can have quantum corrections to the low-energy superpotential [39], as well as independent corrections to the classical moduli space [34]. We study the moduli space seen by a D3-brane probe in the dual geometry.

### 7.1 The structure of the effective superpotential

The  $\mathcal{N} = 1$  superpotential in the  $A_{n-1}$  theories takes the form

$$\begin{aligned} W &= -\text{Tr} \sum_i \frac{h_i}{2} (A_i B_i - B_{i-1} A_{i-1})^2 \\ &= \sum_i \left( h_i \text{Tr} A_i B_i B_{i-1} A_{i-1} - \frac{h_i + h_{i+1}}{2} \text{Tr} (A_i B_i)^2 \right). \end{aligned} \quad (7.1)$$

We will take this theory as the UV completion of the IR physics that we study below, even though (7.1) can be obtained by deformation of an  $\mathcal{N} = 2$  theory. Therefore the couplings  $h_i$  are the fundamental quantities and we will no longer refer to masses  $m_i$ . By analogous arguments to those in section 4, the moduli space of the theory with  $\sum_i h_i^{-1} \neq 0$  is an ALE space, while that for  $\sum_i h_i^{-1} = 0$  is a generalized conifold.

The superpotential (7.1) receives perturbative wavefunction renormalizations, which can be understood as the running of the coupling constants  $h_i$ . There are also additional nonperturbative corrections allowed. These are constrained by the global symmetries and holomorphy. To determine the possible corrections, we need to determine what holomorphic invariants exist. These theories have an  $SU(2)$  global symmetry that is such that the products  $A_i B_i$  or  $B_i A_i$  are invariant. Furthermore, there are several  $U(1)$ s, including a nonanomalous baryon  $U(1)_B$  and the associated anomalous flavor symmetry  $U(1)_F$ . A convenient normalization for these charges is presented in Table 1. In addition, there is an anomalous axial symmetry  $U(1)_A$ . The nonanomalous R-symmetry is a linear combination of  $U(1)_F$  and  $U(1)_A$ .

	$U(N_i)$	$U(N_{i+1})$	$U(1)_B$	$U(1)_F$	$U(1)_A$
$A_i$	$\mathbf{N}_i$	$\overline{\mathbf{N}}_{i+1}$	$\frac{1}{n \prod_i N_i}$	$\frac{1}{n \prod_i N_i}$	0
$B_i$	$\overline{\mathbf{N}}_i$	$\mathbf{N}_{i+1}$	$-\frac{1}{n \prod_i N_i}$	$\frac{1}{n \prod_i N_i}$	0
$\Lambda_i^{b_0^{(i)}}$			0	$(N_{i-1} + N_{i+1}) \left( \frac{1}{n \prod_i N_i} - 1 \right)$	$2N_i - N_{i-1} - N_{i+1}$
$h_i$			0	$-\frac{4}{n \prod_i N_i}$	0

Table 1: The gauge and global symmetry representations of the fields and couplings.

The holomorphic invariants are of the form

$$I_{\zeta_i \xi_i \chi_i} = \text{Tr} \prod_i h_i^{\zeta_i} \left( \Lambda_i^{b_0^{(i)}} \right)^{\xi_i} (A_i B_i)^{\chi_i}, \quad (7.2)$$

where the exponents  $\zeta_i$ ,  $\xi_i$ , and  $\chi_i$  satisfy

$$\begin{aligned} 2\xi_i - \xi_{i-1} - \xi_{i+1} &= 0, \\ 4\zeta_i + (N_{i-1} + N_{i+1}) \left( n \prod_i N_i - 1 \right) \xi_i - 2\chi_i &= 0. \end{aligned} \quad (7.3)$$

We should also include invariants which differ from (7.2) by inequivalent permutations of the fields. The superpotential (7.1) will be renormalized to be of the form

$$\begin{aligned} W = \sum_i & \left[ h_i [\text{Tr} A_i B_i B_{i-1} A_{i-1}] F_i(I_{\zeta_i \xi_i \chi_i}) \right. \\ & \left. - \frac{h_i + h_{i+1}}{2} [\text{Tr} (A_i B_i)^2] G_i(I_{\zeta_i \xi_i \chi_i}) \right], \end{aligned} \quad (7.4)$$

where the  $F_i$  and  $G_i$  are some undetermined functions of the invariants (7.2).

## 7.2 Seiberg Duality

We want to compute the effect of Seiberg dualizing the group  $U(N_1)$ . Therefore we assume that  $b_0^{(1)} = 3N_1 - N_n - N_2 > 0$ , so that  $SU(N_1)$  will confine at some scale  $\Lambda_1$ . The  $U(N_1)$ -invariant degrees of freedom are the mesons

$$Z_{ij} = \begin{pmatrix} A_n A_1 & A_n B_n \\ B_1 A_1 & B_1 B_n \end{pmatrix}. \quad (7.5)$$

If we also introduce the  $U(N_i)$ -invariants  $M_i = A_i B_i$ , we can rewrite (7.1) as

$$\begin{aligned} W = h_1 \text{Tr} & \left[ Z_{11} Z_{22} - \frac{1}{2} (Z_{12}^2 + Z_{21}^2) - \frac{h_n}{2h_1} (Z_{12} - M_{n-1})^2 \right. \\ & \left. - \frac{h_2}{2h_1} (Z_{21} - M_2)^2 + \sum_{i=3}^{n-1} \frac{h_i}{h_1} (M_i - M_{i-1})^2 \right]. \end{aligned} \quad (7.6)$$

For now we will ignore the fact that there are generally nonperturbative renormalizations of (7.6), of the form (7.4).

When  $N_n + N_2 > N_1$ , this theory has a Seiberg dual. The fields  $(A_1, B_1)$ ,  $(A_n, B_n)$  are confined, leaving the mesons  $Z_{ij}$  as low-energy degrees of freedom. The confining gauge group  $U(N_1)$  is replaced by  $U(N_n + N_2 - N_1)$  and additional degrees of freedom

$(a_1, b_1)$ ,  $(a_n, b_n)$  are added. These fields are in the bifundamental representations of  $U(N_n + N_2 - N_1) \times U(N_2)$  and  $U(N_n) \times U(N_n + N_2 - N_1)$ , respectively. We can assemble these into 2-vectors as  $\tilde{q}_i = (a_1, b_n)$ ,  $q_i = (a_n, b_1)$ . Then the superpotential of the dual theory is

$$\begin{aligned} \widetilde{W} = \text{Tr} \left[ y Z_{ij} q_i \tilde{q}_j + h_1 \left( Z_{11} Z_{22} - \frac{1}{2} (Z_{12}^2 + Z_{21}^2) - \frac{h_n}{2h_1} (Z_{12} - M_{n-1})^2 \right. \right. \\ \left. \left. - \frac{h_2}{2h_1} (Z_{21} - M_2)^2 \right) \right] + \dots \end{aligned} \quad (7.7)$$

As the mesons  $Z_{ij}$  are massive, they can be integrated out, leaving the superpotential

$$\begin{aligned} \widetilde{W} = -\frac{y^2}{h_1} \text{Tr} \left[ X_{11} X_{22} - \frac{1}{2} (X_{12}^2 + X_{21}^2) - \frac{\tilde{h}_n}{2\tilde{h}_1} (X_{12} - \widetilde{M}_{n-1})^2 \right. \\ \left. - \frac{\tilde{h}_2}{2\tilde{h}_1} (X_{21} - \widetilde{M}_2)^2 \right] + \dots, \end{aligned} \quad (7.8)$$

where the new mesons are

$$X_{ij} = \begin{pmatrix} a_n a_1 & a_n b_n \\ b_1 a_1 & b_1 b_n \end{pmatrix}, \quad \widetilde{M}_2 = \frac{h_1}{y} M_2, \quad \widetilde{M}_{n-1} = \frac{h_1}{y} M_{n-1}. \quad (7.9)$$

This has the same form as the original superpotential (7.6), except that the coupling constants are shifted as

$$\tilde{h}_1 = -\frac{y^2}{h_1}, \quad \tilde{h}_2 = \frac{y^2 h_2}{h_1(h_1 + h_2)}, \quad \tilde{h}_n = \frac{y^2 h_2}{h_1(h_1 + h_n)}, \quad \text{other } \tilde{h}_i = h_i. \quad (7.10)$$

In this equation, all couplings are meant to be defined at the scale at which the Seiberg duality is performed.

The case that  $n = 2$  is slightly different, as  $Z_{12}$  and  $Z_{21}$  mix. We find that the dual superpotential is

$$\widetilde{W} = -\frac{y^2}{h_1} \text{Tr} \left[ X_{11} X_{22} - \frac{1}{2} \left( 1 + \frac{\tilde{h}_1}{\tilde{h}_2} \right) (X_{12}^2 + X_{21}^2) + \frac{\tilde{h}_1}{\tilde{h}_2} X_{12} X_{21} \right], \quad (7.11)$$

where the shift is now

$$\tilde{h}_1 = -\frac{y^2}{h_1}, \quad \tilde{h}_2 = \frac{y^2(2h_1 + h_2)}{h_1^2}. \quad (7.12)$$

In the theory studied in [21], the bare couplings satisfy  $h_1 + h_2 = 0$ . Therefore the theory has an additional  $SU(2)$  global symmetry in the UV, as reviewed in Section 5.

Nonperturbative corrections will not generate terms that break this symmetry, so  $h_1 + h_2 = 0$  at all scales. As only the ratio  $h_1/h_2$  now appears in the superpotential, the dual superpotential (7.11) is just a rescaling of the original one [40]. Therefore the theory of [21] can be said to be self-dual, at least in this sense, under Seiberg duality [41, 22]. In general, especially for  $n > 2$ , there are families of quartic operators related by Seiberg duality.

After one Seiberg duality, the theory will continue into the IR until the next group confines. At that point, so long as  $N_{i-1} + N_{i+1} - N_i > 0$ , it is possible to Seiberg dualize at the new confining node. We can then follow the theory to the next confining scale and repeat the process. This is a duality cascade. A difference from  $A_1$  to  $A_{n-1}$  is that some  $SU(N_i)$  factors cannot be Seiberg dualized because  $N_{i-1} + N_{i+1} - N_i < 0$ . Also, if  $3N_i - N_{i-1} - N_{i+1} < 0$ , the group  $SU(N_i)$  will be IR free.

Nevertheless, if the  $N_i$  are large enough, there will be a duality cascade over a large range of scales. It is interesting to examine the result of a large number of Seiberg dualities. For simplicity, we will consider  $A_1$  with no restriction on the bare  $h_1 + h_2$ . Seiberg dualities will cycle from node to node, so the result  $h_1^{(s)}/h_2^{(s)}$  of performing  $s$  Seiberg dualities will depend on whether  $s$  is odd or even. If we begin at node 1, then we find that

$$\begin{aligned} s = 2p + 1, \quad \frac{h_1^{(2p+1)}}{h_2^{(2p+1)}} &= \frac{h_2 - (2p+1)(h_1 + h_2)}{h_1 + (2p+1)(h_1 + h_2)}, \\ s = 2p, \quad \frac{h_1^{(2p)}}{h_2^{(2p)}} &= \frac{h_1 + 2p(h_1 + h_2)}{h_2 - 2p(h_1 + h_2)}. \end{aligned} \quad (7.13)$$

The behavior for large  $s$  is independent of  $h_1, h_2$ , namely  $h_1^{(s)}/h_2^{(s)} \rightarrow -1$ ! We also find that  $(h_1^{(s)})^{-1} + (h_2^{(s)})^{-1} \rightarrow 0$  for large  $s$ . Apparently the duality cascade takes the theory to a line of couplings describing the theory of [21]. In the general case, it appears that we also find that  $\lim_{s \rightarrow 0} \sum_i (h_i^{(s)})^{-1} \rightarrow 0$ . Even though the original  $h_i$  are scale-dependent, it turns out that  $\lim_{s \rightarrow 0} \sum_i (h_i^{(s)})^{-1}$  is not.

However, the above analysis of the quartic superpotential is incomplete, due to the nonperturbative corrections to (7.6). Generally the dual superpotential at the scale  $\Lambda_1$  will not be just a quadratic polynomial in the mesons, so it is incorrect to integrate them out via the F-flatness conditions. Instead of (7.8), the dual superpotential should take the form

$$\begin{aligned} W = \sum_i & \left[ \tilde{h}_i [\text{Tr } a_i b_i b_{i-1} a_{i-1}] \tilde{F}_i(\tilde{I}_{\zeta_i \xi_i \chi_i}) \right. \\ & \left. - \frac{\tilde{h}_i + \tilde{h}_{i+1}}{2} [\text{Tr } (a_i b_i)^2] \tilde{G}_i(\tilde{I}_{\zeta_i \xi_i \chi_i}) \right]. \end{aligned} \quad (7.14)$$

Here  $(a_i, b_j)$  refer to the dual variables, but if the Seiberg duality is only performed at node 1, then  $(a_i, b_j) = (A_i, B_i)$  when  $i \neq 1, n$ . In general the functions  $\tilde{F}_i$  and  $\tilde{G}_i$  will

be different from  $F_i$  and  $G_i$ . The mesons can still be massive, but they are charged and interact via various correction terms, so integrating them out to obtain (7.14) is hard.

Nevertheless it is intriguing to consider the possibility that, if all of these corrections could be taken into account, one would still find that the net effect of the duality cascade was to take the theory to the line  $\sum_i h_i^{-1} \rightarrow 0$  in coupling space. This would imply that the theories with different  $h_i$  exhibit some sort of universality in the IR. It is already known that the conformal theories with different  $h_i$  are closely related [7]. Perhaps information from the gravity dual of this theory can be used to shed light on the form of the corrections appearing in (7.6), (7.14), thereby addressing the question of the true IR behavior of these theories.

### 7.3 Infrared Moduli Spaces

As an example of a theory with a quantum correction to the low-energy superpotential, we consider the  $A_{n-1}$  generalization of a theory studied in [21]. We take an  $A_{n-1}$  theory with gauge group  $G = U(N+1) \times U(1)_2 \times \cdots \times U(1)_n$ . This theory describes a single D3-brane probe in the background of  $N$  fractional branes on one of the homology cycles on the  $A_{n-1}$  fiber of a generalized conifold. In the IR, the interacting gauge theory is  $SU(N+1) \subset G$  with 2 flavors. The beta function coefficient is  $b_0^{(1)} = 2N$ , so the interacting theory is asymptotically free and will confine below some scale  $\Lambda_1$ .

Below the scale  $\Lambda_1$ , we should introduce the  $U(N+1)$ -invariant degrees of freedom  $Z_{ij}$  defined in (7.5). The remaining degrees of freedom can be taken to be the mesons  $M_i = A_i B_i$  and the “baryonic” operators

$$x = A_n A_1 \cdots A_{n-1}, \quad y = B_{n-1} \cdots B_1 B_n. \quad (7.15)$$

In terms of these variables, the UV superpotential is given by (7.6). We will call this superpotential  $W_{\text{probe}}$ . Note that  $Z_{11}$  and  $Z_{22}$  are charged under  $U(1)_n \times U(1)_2$ , but the combination  $Z_{11} Z_{22}$  that enters the superpotential (7.6) is invariant. The D-flatness conditions for these  $U(1)$ s are

$$\begin{aligned} |Z_{11}|^2 + |B_2|^2 &= |Z_{22}|^2 + |A_2|^2, \\ |Z_{11}|^2 + |B_{n-1}|^2 &= |Z_{22}|^2 + |A_{n-1}|^2. \end{aligned} \quad (7.16)$$

Together with the other D-flatness conditions, we see that these are a “solution” to

$$xy = Z_{11} Z_{22} M_2 \cdots M_{n-1}. \quad (7.17)$$

This theory also generates a dynamical superpotential [39] in the IR,

$$W_{ADS} = (N-1) \left( \frac{2\Lambda^{3N+1}}{\det Z} \right)^{1/(N-1)}, \quad (7.18)$$

so the complete low-energy effective superpotential is

$$W_{\text{eff}} = W_{\text{probe}} + W_{\text{ADS}}. \quad (7.19)$$

Let us consider the F-flatness condition for  $Z_{11}$ . We find that

$$h_1 Z_{22} - \left( \frac{2\Lambda^{3N+1}}{(\det Z)^N} \right)^{1/(N-1)} Z_{22} = 0, \quad (7.20)$$

with similar equations for the other  $Z_{ij}$ . Together these imply that

$$\det Z = c, \quad c = \left( \frac{(2\Lambda^{3N+1})^{1/(N-1)}}{h_1} \right)^{N/(N-1)} \quad (7.21)$$

and

$$M \cdot \begin{pmatrix} Z_{12} \\ Z_{21} \\ M_2 \\ \vdots \\ M_{n-1} \end{pmatrix} = 0, \quad (7.22)$$

where  $M$  is of the same form as in (4.26).

If  $\sum_i h_i^{-1} \neq 0$ , then the solution to (7.22) is

$$Z_{12} = Z_{21} = M_2 = \cdots = M_{n-1} = z. \quad (7.23)$$

Then (7.21) can be solved for the product

$$Z_{11} Z_{22} = z^2 + c. \quad (7.24)$$

Applying these to the identity (7.17), we find the moduli space

$$xy = z^{n-2}(z^2 + c). \quad (7.25)$$

Therefore the  $A_{n-1}$  singularity has been partially resolved by the dynamically generated superpotential.

When  $\sum_i h_i^{-1} = 0$ , the solution to (7.22) is

$$\begin{aligned} Z_{12} &= z - \gamma_1 t, \quad Z_{21} = z - \gamma_2 t, \quad M_i = z - \gamma_{i+1} t, \\ \sum_i \gamma_i &= 0. \end{aligned} \quad (7.26)$$

Rewriting (7.17), we find that

$$\begin{aligned}
xy &= \left[ (z - \gamma_1 t)(z - \gamma_2 t) + c \right] \prod_{i=3}^n (z - \gamma_i t) \\
&= (z - \tau_+(t))(z - \tau_-(t)) \prod_{i=3}^n (z - \gamma_i t), \\
\tau_{\pm}(t) &= \frac{1}{2}(\gamma_1 + \gamma_2)t \pm \frac{1}{2}\sqrt{(\gamma_1 - \gamma_2)^2 t^2 - 4c}.
\end{aligned} \tag{7.27}$$

This is a generalized conifold which is a monodromic fibration [9] because of the square-root branch cut in  $\tau_{\pm}(t)$ . The only resolvable 2-cycle in the geometry is the one which is wrapped by the fractional branes.

A slight generalization is obtained by wrapping fractional branes around non-adjacent cycles. For example, if we consider  $A_3$  with branes wrapping the cycles corresponding to the first and third nodes, we obtain a theory with superpotential

$$\begin{aligned}
W = \text{Tr} \left[ h_1 (-2Z_{11}Z_{22} + Z_{12}^2 + Z_{21}^2) + h_3 (-2Y_{11}Y_{22} + Y_{12}^2 + Y_{21}^2) \right. \\
\left. + h_2 (Z_{21} - Y_{12})^2 + h_4 (Z_{12} - Y_{21})^2 \right].
\end{aligned} \tag{7.28}$$

The mesons  $Z_{ij}$  and  $Y_{ij}$  are defined in an obvious manner following the conventions of (7.5). Also defining the variables (7.15), the D-terms yield the constraint

$$xy = Z_{11}Z_{22}Y_{11}Y_{22}. \tag{7.29}$$

This theory will develop a dynamical superpotential (7.18) independently for the first and third nodes. The F-flatness conditions for the resulting low-energy effective superpotential imply that

$$\begin{aligned}
\det Z = c_1, \quad \det Y = c_3, \\
M \cdot \begin{pmatrix} Z_{12} \\ Z_{21} \\ Y_{12} \\ Y_{21} \end{pmatrix} = 0.
\end{aligned} \tag{7.30}$$

These can be solved in the usual manner. When  $\sum_i h_i^{-1} \neq 0$ , we recover the ALE space

$$xy = (z^2 + c_1)(z^2 + c_3). \tag{7.31}$$

When  $\sum_i h_i^{-1} = 0$  we find a generalized conifold that is a monodromic fibration

$$xy = (z - \tau_{1,+}(t))(z - \tau_{1,-}(t))(z - \tau_{3,+}(t))(z - \tau_{3,-}(t)), \tag{7.32}$$

where the  $\tau_{\alpha,\pm}$  have square root branch cuts as in (7.27). The generalization to  $A_{n-1}$  is obvious. The only resolvable 2-cycles are the ones which were originally wrapped by fractional branes.

These are essentially the only simple examples. In more general cases, one cannot ignore the fact that fields carry charges under more than one nonabelian gauge group. This leads to nonperturbative corrections to the low-energy superpotential (7.4). We will only attempt to scratch the surface of the corresponding corrections to the classical geometry.

Many salient features of the general case are already present in the example of  $A_{n-1}$  with gauge group

$$G = U(N + M + 1) \times U(N + 1) \times U(1)_3 \cdots \times U(1)_n. \quad (7.33)$$

This theory describes a single probe brane in the presence of fractional branes wrapping two “adjacent” 2-cycles of the ALE space. It is a generalization of the  $A_1$  case of [21]. In the IR, the interacting part of the theory is  $SU(N + M) \times SU(N)$  and both gauge groups are asymptotically free. As we flow to the IR, the group  $SU(N + M)$  will confine first, at the scale  $\mu = \Lambda_1$ . When  $n > 2$ , it is not possible to Seiberg dualize  $SU(N + M)$ . Below  $\Lambda_1$ , the degrees of freedom interacting under  $SU(N)$  are the mesons  $Z_{ij}$  and the pair  $A_2, B_2$ . The superpotential in the UV,  $W_0$ , has the form (7.6).

If the coupling constant,  $g_2$ , of  $SU(N)$  were zero, the only correction to the low-energy superpotential would be of ADS-type,

$$W_{\text{eff.}}(g_2 = 0) = W_0 + (M - 1) \left( \frac{2\Lambda_1^{b_0^{(1)}}}{\text{Tr } \det Z} \right)^{1/(M-1)}. \quad (7.34)$$

When the  $SU(N)$  interaction is turned on, the effective superpotential is no longer so strongly constrained. We will not attempt to determine the precise form of the corrections. On general grounds, we might expect that the functions appearing in (7.4) include functions of  $\text{Tr } \det Z$ . One can then imagine trying to solve the F-flatness conditions by setting

$$\text{Tr } \det Z = c, \quad (7.35)$$

for some constant  $c$ . The rest of the F-flatness conditions will lead to a system of  $n$  nonlinear equations for the  $Z_{ij}$ ,  $M_i$ . Presumably the gauge theory is smart enough to require that this has a solution, at least in principle. Then we will find a moduli space of the form

$$xy = P(y, z) \text{ or } P(y, z, t). \quad (7.36)$$

where  $P(y, \dots)$  is some function. It is not possible to determine the dimension of the moduli space without more information.



## 8 Discussion

Our field theory results should be useful in addressing many aspects of the string theoretic description of these theories. It is known that the relevant geometry is the generalized conifold, but apart from the  $A_1$  case, where solutions are known for  $h_1 = \pm h_2$ , no completely satisfactory explicit classical IIB solutions are known. Apart from the case that  $h_i = h, \forall i$ , no solutions are known when  $\sum_i h_i^{-1} \neq 0$ . Explicit, or at least approximate, solutions would be necessary for the computation of field theory correlation functions from the gravity dual. It is possible that the marginal operators of Section 5 can be useful for generating new solutions from the known ones.

We found that when  $\sum_i h_i^{-1} \neq 0$ , the moduli space of the  $\mathcal{N} = 1$  theory was just the ALE fiber of the generalized conifold. This has an interpretation as the presence of a potential on the probe brane which is generated by the 3-form flux [15]. The probe is transverse to a generalized conifold, but it is sitting at the minimum of this potential, which is just the ALE fiber. It would be interesting to verify this in the IIB duals.

We also found that the existence of a fixed point demanded that we could not generally add F-terms to the original  $\mathcal{N} = 2$  theory when deforming by masses leading to  $\sum_i h_i^{-1} \neq 0$ . These F-terms correspond to complex structure deformations of the ALE, so their absence implies that the 3-form flux presents some sort of obstruction to complex structure deformation in the dual theory. The ALE space admits resolutions, in the form of D-terms in the quiver theory, but since the metric on the ALE is not Ricci-flat, there is no hyperKähler isometry to relate these to complex structure deformation. It is important to understand these results in the dual theory.

It is also important to firmly resolve the issue of general deformations of the  $\mathcal{N} = 2$  theories. We argued in Section 6 that dangerously irrelevant operators  $\Phi^{k+1}$  did not exist for  $k > 2$ . This was based on the  $a$ -theorem, for which no explicit proof exists. In the absence of such a proof, it is possible that our use of the  $a$ -theorem is invalid. Perhaps the candidate fixed points generated by  $\Phi^{k+1}$  do exist, and that the flows away from them violate the  $a$ -theorem. It could be the case that a different central charge plays the role of a function that is monotonically decreasing along the RG flows. In any case, it is important to better understand the  $a$ -theorem in general. It is also interesting to determine whether  $\Phi^3$  is marginally relevant or irrelevant in these theories.

We used an analysis at the critical points of a general deformation  $W(\Phi_i)$  to argue that the moduli space at the fixed point is given by (6.4). This gave a prediction that the coefficient of the mass terms in  $W(\Phi_i)$  should run to  $g_2 \sim W'(t)$  at the fixed point, after all of the irrelevant operators have dropped out of the theory. This implies that the coefficients of the  $\mathcal{N} = 1$  quartic superpotential,  $h_i$ , have a dependence on  $t$ , which is a coordinate on the moduli space of the theory. This is natural, because

the quartic superpotential is computed in the background of the VEVs for the massive fields,  $\langle \Phi_i \rangle \sim t$ . It would be interesting to understand this running of couplings in  $W(\Phi_i)$  better, both in field theory and in the gravity dual. It is certainly related to the position dependence of the flux on the dual geometry.

We also saw that the nonconformal theories obtained by adding fractional branes are quite interesting. We saw that the duality cascade in these theories seems to take theories with  $\sum_i h_i^{-1} \neq 0$  onto a theory with  $\sum_i h_i^{-1} = 0$  in the IR. Correspondingly, the moduli space of scalars is growing an extra complex dimension at the end of the cascade. Quantum corrections to the superpotential prevented us from making a decisive demonstration of this, however. We also saw that quantum corrections lead to interesting deformations of the moduli spaces of the field theory.

It is important to match the gauge theory and gravity descriptions in these cases. Better knowledge of the gravity solution should shed light on the nature of the corrections to the superpotential. The metric structure of the solution should be very close to that of the base of the generalized conifold. Then the solutions for the flux (in particular the 5-form), will be very important for computing the moduli space of a D-brane probe. The differences between the moduli space geometry and that of the original generalized conifold will reflect the corrections to the field theory superpotential.

These corrections would be important for settling the issue of whether the theories do in fact cascade onto theories with  $\sum_i h_i^{-1} = 0$ . It would be interesting to elaborate upon the corresponding behavior of the 3-form flux in the gravity dual.

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